DERIVATIONS AND AUTOMORPHISMS OF $L^1(0, 1)$

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In this paper we investigate the derivations and automorphisms of the radical algebra $L^1(0, 1)$, in which the product of f and g is given by

$$(f * g)(x) = \int_0^x f(x-t)g(t)\,dt.$$

Recall that if X is an algebra, a linear map D is a derivation provided $D(xy) = xD(y)+D(x)y$ for all $x, y \in X$. All automorphisms and derivations in this paper are assumed to be bounded.

In §1 we show that every derivation has the form $Df=xf*\mu$, where $|\mu|[0, t] = \mathcal{O}(1/(1-t))$ as $t \to 1^-$; D is quasinilpotent if and only if μ has no mass at 0. We also determine necessary and sufficient conditions for two derivations to commute. In §2 we prove a converse to the well-known theorem that the exponential of a derivation is an automorphism. We show that if A is an automorphism of a Banach algebra X and the series for $\log A$ converges, then $\log A$ is a derivation on X. In particular, if $\|I-A\| < 1$ or if $I-A$ is quasinilpotent, then A is the exponential of a derivation. §3 is devoted to the study of the automorphism group $\mathcal{A}$ of $L^1(0, 1)$ and its relationship to the space of derivations. We find that $\mathcal{A}_I$, the component of the identity in $\mathcal{A}$, consists of automorphisms of the form $e^{\lambda x}e^q$, where q is a quasinilpotent derivation and λ is a constant. Finally we determine those λ's for which $e^{\lambda x}e^q=e^D$ has a solution D for arbitrary q and also those λ's for which D is unique. We leave open the question of whether $\mathcal{A}_I$ is all of $\mathcal{A}$.

0. **Preliminaries.** Let f^{*n} denote $f*f*\cdots*f$ (n terms). Then $1^{*n}=x^{n-1}/(n-1)!$, the norm of which is $1/n!$. Taking nth roots we find that $\|1^{*n}\|^{1/n} \to 0$, and hence 1 is quasinilpotent. Since 1 generates $L^1(0, 1)$, every element is quasinilpotent and so $L^1(0, 1)$ is a radical algebra. Alternatively, we can observe that any function vanishing a.e. on a neighborhood of 0 is nilpotent, and since these functions are dense in the commutative algebra $L^1(0, 1)$, we again see that $L^1(0, 1)$ is all radical.

A theorem of Titchmarsh [5] asserts that if f_1, f_2, and g are integrable on every interval $(0, T)$ then $f_1 * g=f_2 * g$ on $(0, a)$ if and only if $f_1=f_2$ on $(0, a-b)$, where b is the largest number for which g vanishes a.e. on $(0, b)$.

The closed ideals of $L^1(0, 1)$ are the closed subspaces invariant under the Volterra operator $V: f \to \int_0^x f$. Using Titchmarsh's Theorem, Donoghue [1] has shown that these subspaces have the form $I_a=\{f : f=0 \text{ a.e. on } (0, a)\}$.

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Finally we remark that the zero divisors of $L^1(0, 1)$ coincide with the nilpotent elements and are precisely the members of I_a with $a > 0$.

1. Derivations on $L^1(0, 1)$.

LEMMA 1. *Multiplication by x is a (bounded) derivation on $L^1(0, 1)$.*

Proof.

$$\begin{aligned} x(f * g)(x) &= x\int_0^x f(x-t)g(t)\,dt = \int_0^x (x-t+t)f(x-t)g(t)\,dt \\ &= \int_0^x (x-t)f(x-t)g(t)\,dt + \int_0^x f(x-t)tg(t)\,dt \\ &= (xf * g + f * xg)(x). \end{aligned}$$

THEOREM 2. *The linear operator D is a (bounded) derivation on $L^1(0, 1)$ if and only if there is a measure μ on $[0, 1)$ satisfying* (i) *$Df = xf * \mu$ for $f \in L^1(0, 1)$ and* (ii) *$s\int_0^{1-s} |d\mu|$ remains bounded as $s \to 0^+$. Furthermore, $\|D\| = \sup_{0<s<1} s\int_0^{1-s} |d\mu|$.*

Proof. Suppose μ is given satisfying (i) and (ii). Since $f \to xf$ is a derivation, so is $f \to xf * \mu$, provided the resulting function has well-behaved L^1-norm. However,

$$\begin{aligned} \|xf * \mu\|_1 &\leqq \int_0^1 \int_0^x (x-t)|f(x-t)|\,|d\mu|(t)\,dx = \int_0^1 \int_t^1 (x-t)|f(x-t)|\,dx\,|d\mu|(t) \\ &= \int_0^1 \int_0^{1-t} s|f(s)|\,ds\,|d\mu|(t) = \int_0^1 |f(s)|\left(s\int_0^{1-s} |d\mu|\right) ds \leqq K\|f\|_1, \end{aligned}$$

by (ii). The norm of D is the supremum of $|\int_0^1 (xf * \mu)g|$ where $\|f\|_1 = 1 = \|g\|_\infty$. A change of variables similar to the above computation shows that

$$\int_0^1 (xf * \mu)g = \int_0^1 \left(s\int_0^{1-s} g(s+t)\,d\mu(t)\right) f(s)\,ds.$$

Therefore $\|D\|$ is the supremum of $\|s\int_0^{1-s} g(s+t)\,d\mu(t)\|_\infty$ for $|g| \leqq 1$, $0 < s < 1$. Clearly then $\|D\| = \sup s\int_0^{1-s} |d\mu|$.

Conversely let D be a derivation on $L^1(0, 1)$. Since $1^{*n} = x^{n-1}/(n-1)!$ it follows that (1) $Dx^n = n!\,D(1^{*(n+1)}) = (n+1)!\,1^{*n} * D1 = (n+1)nx^{n-1} * D1$. Therefore D is determined by the function $g = D1$. From (1) we have that (2) $Dp = (xp)'' * g$ for polynomials p, with the convention that x'' is the unit mass at 0. In particular it is clear that (2) holds for C^2 functions which vanish identically near 0. Call this class of functions F.

Let φ_n be a C^2 approximate identity for $L^1(0, 1)$ such that $\varphi_n(0) = \varphi_n'(0) = \varphi_n''(0) = 0$. Put $g_n = g * \varphi_n$. Then g_n is in C^2 and $g_n'' = g * \varphi_n''$. We may assume that $g_n \to g$ a.e.

For f in F integration by parts twice gives $(xf)'' * g_n = xf * g_n''$, since both f and g_n vanish to second order at 0.

For each $\delta \in (0, 1)$ let F_δ be those C^2 functions which vanish on $[0, \delta]$. Then on F_δ the transformation $f \to f * g_n''$ has norm at most $\|D\|/\delta$, since

$$\|f * g_n''\| = \|f'' * g_n\| = \|f'' * g * \varphi_n\| \leqq \|f'' * g\| = \|(xf/x)'' * g\|$$
$$= \|D(f/x)\| \leqq \|D\| \, \|f/x\| \leqq (\|D\|/\delta)\|f\|.$$

This means that $\int_0^{1-\delta} |g_n''| \leqq \|D\|/\delta$.

Now let $\delta_n = 2^{-n}$ and for each n choose a net of the g_k'' converging in the weak-$C\,[0, 1]$ topology to a measure μ_n on $[0, 1-\delta_n]$ in such a way that the net for $(n+1)$ is a subnet of the net for n. That is, μ_{n+1} extends μ_n. Thus we obtain a measure μ on $[0, 1)$ such that $\int_0^{1-\delta_n} |d\mu| \leqq \|D\|/\delta_n$, and hence $\int_0^{1-\delta} |d\mu| \leqq 2\|D\|/\delta$ for $0<\delta<1$. Furthermore if f is continuous on $[0, 1]$ and vanishes identically near 1, there is a net of g_k'' so that $\int_0^1 fg_k'' \to \int_0^1 f\, d\mu$.

By the first half of the theorem the measure μ defines a derivation D_μ on $L^1(0, 1)$: $D_\mu f = xf * \mu$. To show that $D_\mu = D$ we need only show that $D_\mu 1 = D1 = g$. Take $0<x<1$ and let $f(t) = (x-t)$ for $0 \leqq t \leqq x$ and $f(t) = 0$ for $x \leqq t \leqq 1$; f is continuous and vanishes identically near 1. Therefore

$$D_\mu 1(x) = \int_0^x (x-t)\, d\mu(t) = \int_0^1 f(t)\, d\mu(t) = \lim \int_0^1 f(t) g_k''(t)\, dt$$
$$= \lim \int_0^x (x-t) g_k''(t)\, dt = \lim\, (x * g_k'')(x)$$
$$= \lim\, (1 * 1 * g_k'')(x) = \lim g_k(x) = g(x) \text{ a.e.}$$

Definition. Let d denote the derivation multiplication by x: $(df)(x) = xf(x)$.

Remarks. 1. By splitting off the mass at 0 in Theorem 2 we can write any derivation D as $D = \lambda d + q$, where $qf = xf * \mu_0$ and μ_0 has no mass at 0. This decomposition is convenient because, as we prove below, D is quasinilpotent if and only if $\lambda = 0$.

2. Letting $s \to 1$ in the expression for $\|D\|$ in Theorem 2, we see that the map $\lambda d + q \to \lambda d$ is linear and of norm 1.

Before proving Remark 1, we make two definitions which will be useful for certain computations.

Definition. If G is an operator on $L^1(0, 1)$, we define G to be *positive* if $Gf \geqq 0$ a.e. whenever $f \geqq 0$ a.e. An operator F *is dominated by* the positive operator G ($F \ll G$ or $F = \mathcal{O}(G)$) if $|Ff| \leqq G|f|$ a.e. for all f.

We note that if $F \ll G$, then $\|F\| \leqq \|G\|$. Furthermore if $F \ll G$, then $F^2 \ll G^2$. Indeed, $|F^2 f| \leqq G|Ff| \leqq GG|f|$. Similarly $F^n \ll G^n$. A consequence of this is that if $F \ll G$ and G is quasinilpotent, then F must be quasinilpotent also.

Definition. If $Df = xf * \mu$, define $|D|$ to be the derivation $|D|f = xf * |\mu|$. Notice that $D \ll |D|$ and, by Remark 1, D is quasinilpotent if and only if $|D|$ is quasinilpotent.

Proof of Remark 1. If $D = \lambda d + q$ and $\lambda \neq 0$, then $D^n = \lambda^n d^n +$ terms involving q. Let $f_k(x) = k$ for $1-(1/k) < x < 1$ and $f_k = 0$ otherwise. Clearly $\|f_k\| = 1$. Also

$\int_0^1 D^n f_k \to \lambda^n$ as $k \to \infty$, because the terms involving q involve convolutions with measures which are small near 0 and f_k lives near 1 only. Therefore $\|D^n\| \geqq |\lambda|^n$ and lim $\|D^n\|^{1/n} \geqq |\lambda| > 0$.

If $\lambda = 0$ we have $Df = qf = xf * \mu_0$, where μ_0 has no mass at 0. Write $\mu_0 = \mu_1 + \mu_2$ where $\mu_1 = \mu_0$ restricted to $[0, \frac{1}{2})$ and $\mu_2 = \mu_0$ restricted to $[\frac{1}{2}, 1)$. Then $q = q_1 + q_2$, where $q_i f = xf * \mu_i$. Let Q_1 be the operator $Q_1 f = f * |\mu|_1$. In the commutative algebra of convolution operators Q_1 is quasinilpotent because it is the limit of the nilpotent operators obtained by restricting $|\mu_1|$ to $(\varepsilon, 1)$, $\varepsilon \to 0$. Since q_1 is clearly dominated by Q_1, we have that q_1 is quasinilpotent. For convenience choose $\varepsilon_t \downarrow 0$ as $t \to \infty$ such that $\|q_1^n\|^{1/n} \leqq \varepsilon_n$.

Now q_2 involves translation by $\frac{1}{2}$, so $q^n = (q_1 + q_2)^n = q_1^n + \sum_{k=0}^{n-1} q_1^k q_2 q_1^{n-k-1}$, since all the terms with two or more q_2's vanish. Let $K = \|q_2\| \max_k \|q_1^k\| < \infty$. Then $\|q^n\| \leqq \varepsilon_n^n + nK\varepsilon_{n/3}^{n/3}$, since at least one of k and $n-k-1$ is $\geqq (n-1)/2 > n/3$. Thus q is quasinilpotent.

It should be noticed that the notion of dominance allowed us to prove for Remark 1 that the sum of two quasinilpotent derivatives is again quasinilpotent, almost as though the derivations commuted with each other. We shall use this kind of argument frequently in §3.

We now determine those pairs of derivations D_1 and D_2 which commute. Let $D_k f = xf * \mu_k$, by Theorem 2. Then

$$D_1 D_2 f = x(xf * \mu_2) * \mu_1 = x^2 f * \mu_2 * \mu_1 + xf * x\mu_2 * \mu_1$$

and

$$D_2 D_1 f = x(xf * \mu_1) * \mu_2 = x^2 f * \mu_1 * \mu_2 + xf * x\mu_1 * \mu_2.$$

Thus $D_1 D_2 = D_2 D_1$ if and only if $xf * (x\mu_1 * \mu_2) = xf * (\mu_1 * x\mu_2)$ for all f. That is (3) $D_1 D_2 = D_2 D_1$ if and only if $x\mu_1 * \mu_2 = \mu_1 * x\mu_2$.

LEMMA 3. *Let f and g belong to $L^1(0, \infty)$ and suppose that $xf * g = f * xg$ on* $(0, 1)$. *Then $x^n f * g = f * x^n g$ on* $(0, 1)$ *for all positive integers n.*

Proof. Let b be the largest number such that g vanishes a.e. on $(0, b)$. By induction assume that $x^n f * g = f * x^n g$ on $(0, 1)$ for some $n \geqq 1$. Convolving with g gives $x^n f * g * g = f * x^n g * g$ on $(0, 1+b)$. Multiplying by x and using Lemma 1 we obtain

$$\begin{aligned} x^{n+1} f * g * g + x^n f * xg * g + x^n f * g * xg \\ = xf * x^n g * g + f * x^{n+1} g * g + f * x^n g * xg \end{aligned}$$

on $(0, 1+b)$. Using commutivity of $*$ and the hypothesis we obtain

$$x^{n+1} f * g * g + 2x^n f * g * xg = 2f * x^n g * xg + f * x^{n+1} g * g \text{ on } (0, 1+b).$$

However, since $x^n f * g = f * x^n g$ on $(0, 1)$, $2(x^n f * g) * g = 2(f * x^n g) * g$ on $(0, 1+b)$ and hence $x^{n+1} f * g * g = f * x^{n+1} g * g$ on $(0, 1+b)$, and so $x^{n+1} f * g = f * x^{n+1} g$ on $(0, 1)$, completing the induction.

LEMMA 4. *Let g be continuous on* $[0, 1]$ *and let b be the largest number such that g vanishes a.e. on* $(0, b)$. *Then the only continuous solutions to $xf*g=f*xg$ are of the form* (A) $f=cg$ *on* $[0, 1-b]$, f *arbitrary on* $(1-b, 1]$, *where c is a constant.*

Proof. It is obvious that functions of the form (A) satisfy $xf*g=f*xg$. Conversely, suppose $xf*g=f*xg$. Then by Lemma 3 we have that $x^nf*g=f*x^ng$ for all n and hence $Pf*g=f*Pg$ for polynomials P and therefore for bounded measurable functions P. If we write this equation as

$$\int_0^x P(t)f(t)g(x-t)\,dt = \int_0^x f(x-t)P(t)g(t)\,dt$$

and let (for $a\leqq x)P(t)=1$ for $0\leqq t\leqq a$ and $P(t)=0$ elsewhere, we have

$$\int_0^a f(t)g(x-t)\,dt = \int_0^a f(x-t)g(t)\,dt.$$

This holds for all $a\leqq x$. Differentiating with respect to a we obtain $f(a)g(x-a)=f(x-a)g(a)$ for all x and all $a\leqq x$. If we now let $a\to b^+$ through values for which $g(a)\neq 0$, we obtain $f(x-b)=cg(x-b)$, with c the common value of $f(a)/g(a)$. That is, for $0\leqq t\leqq 1-b$, $f(t)=cg(t)$.

With these lemmas we are now able to determine when two derivations commute.

THEOREM 5. *Let $D_if=xf*\mu_i$ $(i=1, 2)$ be derivations on $L^1(0, 1)$. Then $D_1D_2=D_2D_1$ if and only if $\mu_1=c\mu_2$ on* $[0, 1-b)$, *where c is a constant and b is the largest number such that* $|\mu_2|\,[0, b)=0$.

Proof. From (3) we know that $D_1D_2=D_2D_1$ if and only if $x\mu_1*\mu_2=\mu_1*x\mu_2$. This holds if and only if $x*x\mu_1*\mu_2*x=x*\mu_1*x\mu_2*x$, or equivalently

$$x*x\mu_1*\mu_2*x+x^2*\mu_1*x*\mu_2 = x*\mu_1*x\mu_2*x+x^2*\mu_1*x*\mu_2,$$

which is

$$(x*x\mu_1+x^2*\mu_1)*(x*\mu_2) = (x*\mu_1)*(x\mu_2*x+x^2*\mu_2),$$

which is $x(x*\mu_1)*(x*\mu_2)=(x*\mu_1)*x(x*\mu_2)$. Repeating the argument with $x*\mu_i$ replacing μ_i we obtain that $D_1D_2=D_2D_1$ if and only if

$$x(x*x*\mu_1)*(x*x*\mu_2) = (x*x*\mu_1)*x(x*x*\mu_2).$$

Now $x*\mu_i=x\cdot 1*\mu_i=D_i1\in L^1(0, 1)$ and therefore $x*x*\mu_i$ is continuous on $[0, 1]$. By Lemma 4, $D_1D_2=D_2D_1$ if and only if $x*x*\mu_1=cx*x*\mu_2$ on $[0, 1-b]$, which by differentiation is equivalent to $\mu_1=c\mu_2$ on $[0, 1-b)$.

We remark that the computations immediately preceding (3) show that $(D_1D_2-D_2D_1)(f)=xf*(\mu_1*x\mu_2-x\mu_1*\mu_2)$, when D_i is given by $D_if=xf*\mu_i$. Two observations can be made from this equation. The first is that the measure $\mu_1*x\mu_2-x\mu_1*\mu_2$ clearly has no mass at 0 and, by Remark 1, $D_1D_2-D_2D_1$ is therefore quasinilpotent. Secondly if $Df=xf*\mu$, then $(dD-Dd)f=xf*x\mu$.

2. The logarithm of an automorphism. It is well known that for any Banach algebra the exponential of a bounded derivation is a bounded automorphism. In general the converse is not true, since, for example, a semisimple commutative Banach algebra has no nonzero bounded derivations [4], but there may be many automorphisms. The following theorem asserts that an automorphism near enough to the identity is the exponential of a derivation. In the next section we shall apply this theorem to automorphisms on $L^1(0, 1)$.

THEOREM 6. *Let A be an automorphism of a Banach algebra X and put $B=A-I$. If the series $\log(I+B)=B-B^2/2+B^3/3-\cdots$ converges in the norm, then it defines a bounded derivation on X. In particular, if either $\|A-I\|<1$ or if $A-I$ is quasi-nilpotent, then A is the exponential of a (bounded) derivation.*

Proof. Put $D=\log A=B-B^2/2+B^3/3-\cdots$. We are to show that for all x and y in X, $D(xy)-(Dx)y-x(Dy)=0$.

Observe that

$$xy+B(xy) = A(xy) = Ax\cdot Ay = (I+B)x\cdot(I+B)y$$
$$= xy+(Bx)y+x(By)+(Bx)(By);$$

hence $B(xy)=(Bx)y+x(By)+(Bx)(By)$. It follows that $B^k(xy) = (B^kx)y+x(B^ky)+$terms of the form $(B^mx)(B^ny)$ with both m and $n \geqq 1$. Thus $D(xy)-(Dx)y-x(Dy)=\sum a_{m,n}(B^mx)(B^ny)=\sum(x, y)$. We shall show that $\sum(x, y)\equiv 0$ by showing that all the $a_{m,n}=0$.

By induction on $m+n$ let us assume that $a_{m,n}=0$ for all $m+n<p$, for some $p\geqq 2$. Since m and n are always $\geqq 1$ the case $p=2$ is vacuously true. We shall now choose B, x, and y so that $\sum(x, y)$ reduces to a finite sum of linearly independent elements with coefficients $a_{m,n}$ $(m+n=p)$. This will complete the induction.

Let X be the ring of real polynomials in two variables x and y, modulo the ideal spanned by the monomials x^my^n, for $m+n>p+2$. Let D be the derivation on X given by $D(F)=x^2\,\partial F/\partial x+y^2\,\partial F/\partial y$. Clearly, D is a nilpotent derivation $(D^{p+2}\equiv 0)$. Let $A=e^D$. Since D is nilpotent, so is $B=A-I=e^D-I$, and hence $D=\log A=B-B^2/2+B^3/3-\cdots$.

From $D^kx=k!\,x^{k+1}$ and $B^m=D^m+$higher powers of D it follows easily that $B^mx=m!\,x^{m+1}+$higher powers of x, and similarly $B^ny=n!\,y^{n+1}+$higher powers of y. Therefore,

$$\sum(x, y) = \sum_{m,n\geqq 1} a_{m,n}B^mxB^ny = \sum_{m+n\geqq p} a_{m,n}B^mxB^ny = \sum_{m+n=p} a_{m,n}m!\,n!\,x^{m+1}y^{n+1},$$

since $a_{m,n}=0$ for $m+n<p$ by induction and $x^iy^j=0$ for $i+j>p+2$. However, $\sum(x, y)=D(xy)-x(Dy)-(Dx)y=0$, since D is a derivation; therefore $a_{m,n}=0$ for $m+n=p$, and the induction is complete.

COROLLARY 7. *If e^T is an automorphism of X and if $\|T\|<\log 2$, then T is a derivation.*

Proof. Since $\|e^T - I\| \leqq e^{\|T\|} - 1 < 2 - 1 = 1$, the series for $\log e^T$ converges to T.

COROLLARY 8. *If A is an automorphism of a semisimple commutative Banach algebra, then $A = I$ or else $\|A - I\| \geqq 1$.*

Proof. There are no nonzero bounded derivations on such an algebra.

REMARKS. 3. In Corollary 7 some condition on T is necessary, since $e^{2\pi i} = I$, but $2\pi i$ is not a derivation unless all products in X are zero.

4. The convergence of the logarithmic series (i.e., the spectral radius of $I - A$ less than 1) is the best general hypothesis we can make. In §3 we shall see that many automorphisms with large spectral radius are nevertheless exponentials of derivations. Moreover, the following examples exhibit (i) an automorphism satisfying $\|I - A\| = 1 + \varepsilon$ for $\varepsilon > 0$ and failing to have a logarithm at all, and (ii) an automorphism with a logarithm but with no derivation as logarithm.

(i) For $\varepsilon > 0$ let X be the space of analytic functions on $\frac{1}{2}\varepsilon < |z| < \varepsilon$ which are in $L^2(dx\, dy)$. X is a Banach space and we make it into a Banach algebra trivially by defining all products to be zero. Let A be the automorphism of X given by $f(z) \to zf(z)$. Since $I - A$ is multiplication by $1 - z$, clearly $\|I - A\| = 1 + \varepsilon$. However, it is known [2] that A has no square root, and hence no logarithm.

(ii) Let $X = L^1(0, \infty)$ with convolution. X is a commutative semisimple Banach algebra and hence has no bounded derivations, except 0. However, the automorphism $f(x) \to e^{2\pi i x} f(x)$ has as logarithm the transformation

$$f(x) \to 2\pi i(x - [x])f(x),$$

where $[x]$ is the integer part of x.

5. From $B(xy) = (Bx)y + x(By) + (Bx)(By)$ in Theorem 6, one can compute B^k by induction and obtain $B^k(xy) = (B^k x)y + x(B^k y) + \sum'_{m,n} a^{(k)}_{m,n} B^m x B^n y$, where $\sum'$ extends over (m, n): $1 \leqq m, n \leqq k \leqq m + n$ and $a^{(k)}_{m,n} = k!/[(n+m-k)!\,(k-n)!\,(k-m)!]$. Then in the proof of Theorem 6, $a_{m,n} = \sum''_k (-1)^k a^{(k)}_{m,n}/k$, where $\sum''$ extends over k: $m, n \leqq k \leqq m + n$.

We thank the referee for the following proof that $a_{m,n} \equiv 0$. We may assume $1 \leqq m \leqq n$, since $a_{m,n} = a_{n,m}$. Then by an easy manipulation of factorials

$$\begin{aligned} a_{m,n} &= \sum_{k=n}^{n+m} (-1)^k a^{(k)}_{m,n}/k = \frac{(-1)^n}{m!} \sum_{t=0}^{m} (-1)^t \binom{m}{t} \frac{(t+n-1)!}{(t+n-m)!} \\ &= \frac{(-1)^{n+m}}{m!} \Delta^m P(n - m + 1), \end{aligned}$$

where Δ is the difference operator $\Delta f(x) = f(x+1) - f(x)$ and P is the polynomial $P(x) = x(x+1) \cdots (x + m - 2)$. Since P has degree $m - 1$, $\Delta^m P(x) \equiv 0$. (The case $m = 1$ gives $P \equiv 1$, since the product is empty.)

3. **Automorphisms of** $L^1(0, 1)$. Let $\mathscr{A}$ be the group of automorphisms of $L^1(0, 1)$, and let $\mathscr{A}$ have the uniform topology ($A_n \to A$ means $\|A_n - A\| \to 0$). Denote by $\mathscr{A}_I$ the connected component of the identity.

THEOREM 9. *The automorphism $A \in \mathscr{A}_I$ if and only if there is a constant λ and a quasinilpotent derivation q such that $A = e^{\lambda d}e^q$, where d is again the derivation $f \to xf$.*

LEMMA 10. *If q is a quasinilpotent derivation on $L^1(0, 1)$ and λ is a constant, then there exists q', a quasinilpotent derivation on $L^1(0, 1)$, such that $e^{\lambda d+q} = e^{\lambda d}e^{q'}$.*

Proof. By Theorem 2 and Remark 1, q is given by $qf = xf * \mu$, where μ has no mass at 0. Recall that $|q|f = xf * |\mu|$ defines a quasinilpotent derivation $|q|$ which dominates q ($q = \mathcal{O}(|q|)$). Therefore,

$$(\lambda d+q)^n = \lambda^n d^n + \mathcal{O}\left(\sum_{m<n} \binom{n}{m} |\lambda|^m |q|^{n-m}\right) = \lambda^n d^n + \mathcal{O}((|\lambda| + |q|)^n - |\lambda|^n).$$

Therefore

$$e^{\lambda d+q} = e^{\lambda d} + \mathcal{O}(e^{|\lambda|+|q|} - e^{|\lambda|}) = e^{\lambda d} + \mathcal{O}(e^{|\lambda|}(e^{|q|} - 1)).$$

Hence $e^{-\lambda d}e^{\lambda d+q} = I + \mathcal{O}(e^{2|\lambda|}(e^{|q|} - 1)) = I +$ a quasinilpotent operator, since $|q|$ is quasinilpotent. Since $e^{-\lambda d}e^{\lambda d+q}$ is an automorphism, Theorem 6 gives $e^{-\lambda d}e^{\lambda d+q} = e^{q'}$, where q' is a derivation; q' is clearly quasinilpotent.

Proof of Theorem 9. Let E be the class of all automorphisms of the form $e^{\lambda d}e^q$, where λ is a constant and q is a quasinilpotent derivation. Clearly E is connected, since $e^{t\lambda d}e^{tq}$, $0 \leqq t \leqq 1$, connects the identity with $e^{\lambda d}e^q$. By Theorem 6 and Lemma 10, E contains a neighborhood of I. Furthermore, A_1 and A_2 in E imply $A_1A_2 \in E$, as follows: $A_1A_2 = e^{\lambda_1 d}e^{q_1}e^{\lambda_2 d}e^{q_2} = e^{(\lambda_1+\lambda_2)d}[e^{-\lambda_2 d}e^{q_1}e^{\lambda_2 d}]e^{q_2}$, and the expression inside the brackets is equal to $\exp(e^{-\lambda_2 d}q_1e^{\lambda_2 d})$. We have $A_1A_2 = e^{(\lambda_1+\lambda_2)d}e^{q_3}e^{q_2}$, where q_3 is the quasinilpotent derivation $e^{-\lambda_2 d}q_1e^{\lambda_2 d}$. Now

$$\begin{aligned} e^{q_3}e^{q_2} &= [1 + \mathcal{O}(e^{|q_3|} - 1)][1 + \mathcal{O}(e^{|q_2|} - 1)] = [1 + \mathcal{O}(e^{|q_2|+|q_3|} - 1)][1 + \mathcal{O}(e^{|q_2|+|q_3|} - 1)] \\ &= 1 + \mathcal{O}(e^{2(|q_2|+|q_3|)} - 1) = 1 + q_4, \end{aligned}$$

where q_4 is a quasinilpotent operator. Therefore, $e^{q_3}e^{q_2} = 1 + q_4 = e^{q_5}$, where q_5 is a quasinilpotent derivation, as in the proof of Lemma 10. Thus, E is closed under multiplication.

Now suppose $A = e^{\lambda d}e^q \in E$. Then

$$A^{-1} = e^{-q}e^{-\lambda d} = e^{-\lambda d}(e^{\lambda d}e^{-q}e^{-\lambda d}) = e^{-\lambda d}\exp(-e^{\lambda d}qe^{-\lambda d}) \in E,$$

since $e^{\lambda d}(-q)e^{-\lambda d}$ is a quasinilpotent derivation. Therefore E is an open connected subgroup of $\mathscr{A}$ containing I. Q.E.D.

We now investigate the existence and uniqueness of solutions to $A = e^D$, where A is to be a given automorphism and the derivation D is to be found. Of course, A must be required to be in $\mathscr{A}_I$.

LEMMA 11. *Let λ be a constant and q a quasinilpotent derivation on $L^1(0, 1)$. Then* $\mathrm{sp}\,(e^{\lambda d}e^q) \subseteq \mathrm{sp}\,(e^{\lambda d}) = \{e^{\lambda x} : 0 \leqq x \leqq 1\}$ *and* $\mathrm{sp}\,(\lambda d+q) \subseteq \mathrm{sp}\,(\lambda d) = \{\lambda x : 0 \leqq x \leqq 1\}$, *where* $\mathrm{sp}\,(T)$ *denotes the spectrum of the operator T.*

Proof. The equalities are obvious, since $e^{\lambda d}$ is multiplication by $e^{\lambda x}$ and λd is multiplication by λx.

If $\alpha \notin \operatorname{sp}(e^{\lambda d})$, then $|\alpha - e^{\lambda x}|^{-1} < K < \infty$ for $0 \leqq x \leqq 1$. Then

$$\begin{aligned}\alpha - e^{\lambda d}e^q &= (\alpha - e^{\lambda d}) - e^{\lambda d}(e^q - 1) = (\alpha - e^{\lambda d})[1 - (\alpha - e^{\lambda d})^{-1}e^{\lambda d}(e^q - 1)] \\ &= (\alpha - e^{\lambda d})[1 + \mathcal{O}(Ke^{|\lambda|}(e^{|q|} - 1))],\end{aligned}$$

which is invertible, since $e^{|q|} - 1$ is quasinilpotent.

Similarly, if $\alpha \notin \operatorname{sp}(\lambda d)$, then $|\alpha - \lambda d|^{-1} < K$ for $0 \leqq x \leqq 1$, and $\alpha - (\lambda d + q) = (\alpha - \lambda d) - q = (\alpha - \lambda d)[1 - (\alpha - \lambda d)^{-1}q] = (\alpha - \lambda d)[1 + \mathcal{O}(K|q|)]$, which is invertible because $|q|$ is quasinilpotent.

LEMMA 12. *For $i = 1, 2$, let $D_i f = xf * \mu_i$ be derivations on $L^1(0, 1)$ and let λ_i be the mass of μ_i at 0. If $e^{D_1} = e^{D_2}$, then $\lambda_1 = \lambda_2$.*

Proof. We can regard the D_i as derivations on $L^1(0, \varepsilon)$, simply by restricting all functions to $(0, \varepsilon)$. It is still true that $e^{D_1} = e^{D_2}$. If ε is small enough, the norms of the D_i will be small, and so will the norms of $e^{D_i} - I$. Then the logarithmic series will converge and give $D_1 = -\sum (I - e^{D_1})^n/n = -\sum (I - e^{D_2})^n/n = D_2$ on $L^1(0, \varepsilon)$. This means that $\mu_1 = \mu_2$ on $[0, \varepsilon)$. In particular, taking the mass at 0, we have $\lambda_1 = \lambda_2$.

LEMMA 13. *The representation of an automorphism as $e^{\lambda d}e^q$, as in* Theorem 9, *is unique. That is, if $e^{\lambda_1 d}e^{q_1} = e^{\lambda_2 d}e^{q_2}$, then $\lambda_1 = \lambda_2$ and $q_1 = q_2$.*

Proof. From $e^{\lambda_1 d}e^{q_1} = e^{\lambda_2 d}e^{q_2}$ we obtain $e^{(\lambda_1 - \lambda_2)d} = e^{q_2}e^{-q_1}$. As in the proof of Theorem 9, $e^{q_2}e^{-q_1} = e^{q_3}$, where q_3 is a quasinilpotent derivation. By Lemma 12 $\lambda_1 - \lambda_2 = 0$. We therefore have $e^{q_1} = e^{q_2}$. Since q_1 is quasinilpotent, so is $I - e^{q_1}$, and we have $q_1 = -\sum (I - e^{q_1})^n/n = -\sum (I - e^{q_2})^n/n = q_2$.

THEOREM 14. *Let $A = e^{\lambda d}e^q$, where λ is a constant and q is a quasinilpotent derivation on $L^1(0, 1)$. Suppose $\lambda \notin \Lambda = \{$pure imaginary numbers of modulus $\geqq 2\pi\}$. Then there exists a unique derivation D on $L^1(0, 1)$ such that $A = e^D$. Furthermore, $D = \lambda d + q'$, where q' is a quasinilpotent derivation.*

Proof. By Lemma 11 the spectrum $\operatorname{sp}(A)$ lies on the arc $\boldsymbol{a} = \{e^{\lambda x} : 0 \leqq x \leqq 1\}$, which does not contain 0 and by the hypothesis does not separate the plane. Let γ cut the plane from 0 to ∞, avoiding $\boldsymbol{a}$, and define $\log z$ off γ so that $\log 1 = 0$. Let Γ be a simple smooth closed curve enclosing $\boldsymbol{a}$ and not meeting γ. Then a logarithm of A may be defined by

$$L = L(A) = (1/2\pi i) \int_\Gamma (z - A)^{-1} \log z \, dz. \quad \text{(See [3].)} \tag{4}$$

We shall show that L is a derivation on $L^1(0, 1)$. Put $A(t) = e^{t\lambda d}e^q$ and let $L(t)$ be the logarithm of $A(t)$ defined by (4). Although t varies, Γ and the branch of logarithm remain fixed. Note that $\operatorname{sp}(A(t)) \subseteq \boldsymbol{a}$ for $0 \leqq t \leqq 1$, and also that $L(t)$ is analytic in t. Also notice that, for small t, $\operatorname{sp}(A(t))$ lies in $|z - 1| < \varepsilon < 1$ and by

Theorem 6 the series $-\sum (I-A(t))^n/n$ converges to a derivation $D(t)$. Now if ε is small enough, the disc $|z-1| \leqq \varepsilon$ does not meet γ, and hence we can deform Γ in the complement of γ to become the circle $|z-1|=\varepsilon$ in such a way that throughout the deformation Γ always encloses the disc $|z-1|<\varepsilon$. Thus, for small t, $L(t)$ is given by (4) with integration over the circle $|z-1|=\varepsilon$. Expand

$$(z-A(t))^{-1} = -(1-z)^{-1} \sum \left(\frac{I-A(t)}{1-z}\right)^n,$$

which converges uniformly for $|z-1|=\varepsilon$ and for small t. Term-by-term integration then gives $L(t) = -\sum (I-A(t))^n/n = D(t)$.

Therefore we have that for each f and g in $L^1(0, 1)$, $L(t)(f*g)-(L(t)f)*g-f*(L(t)g)$ is analytic in t and vanishes for all small t. Therefore it vanishes for all t and in particular for $t=1$. That is, $\log A = L$ is a derivation.

Now write $L=\lambda' d+q'$, where λ' is a constant and q' is a quasinilpotent derivation. To see that $\lambda'=\lambda$, write $e^{\lambda' d+q'}=e^{\lambda' d}e^{q''}$ by Lemma 10. Then we have $e^{\lambda d}e^q = A = e^{\lambda' d}e^{q''}$, and by Lemma 13, $\lambda'=\lambda$.

To see that L is unique, suppose that L_1 is another derivation on $L^1(0, 1)$ and $e^{L_1}=A=e^L$. By Lemma 6 we may write $L_1=\lambda d+q_1$. Put $A_1(t)=e^{tL_1}$. The formula (4) with $A_1(t)$ replacing A defines an analytic function $L(A_1(t))$ of t, since the spectra of $A_1(t)$ lie on $\boldsymbol{a}$ for $0\leqq t\leqq 1$. For small t we deform Γ as before and compute directly that $L(A_1(t))=tL_1$. We obtain $L=L_1$ by putting $t=1$.

Definition. Let $\mathscr{D}'$ be the class of derivations of the form $D=\lambda d+q$, where q is quasinilpotent and $\lambda \notin \Lambda$.

Corollary 15. *If D_1 and D_2 belong to $\mathscr{D}'$ and $e^{D_1}e^{D_2}=e^{D_2}e^{D_1}$, then $D_1D_2=D_2D_1$.*

Proof. From the hypothesis we obtain $e^{D_1}=e^{D_2}e^{D_1}e^{-D_2}=\exp(e^{D_2}D_1e^{-D_2})$. Since both D_1 and $e^{D_2}D_1e^{-D_2}$ are in $\mathscr{D}'$, we conclude by the uniqueness part of Theorem 14 that $D_1=e^{D_2}D_1e^{-D_2}$. Therefore $D_1^n=e^{D_2}D_1^n e^{-D_2}$ for all n, and so $e^{tD_1}e^{D_2}=e^{D_2}e^{tD_1}$ for all t. Thus, D_2 commutes with e^{tD_1}, since as before $e^{D_2}=\exp(e^{-tD_1}D_2e^{tD_1})$. Now we write $e^{tD_1}D_2=D_2e^{tD_1}$ and differentiate at $t=0$ to obtain $D_1D_2=D_2D_1$.

Remark 6. If A is any automorphism on $L^1(0, 1)$ and $Ae^d=e^dA$, then $A=e^{td}$ for some t. For $e^d=A^{-1}e^dA$ implies $d=A^{-1}dA$ as in Corollary 15, since it is easily verified that $A^{-1}dA\in\mathscr{D}'$. From $Ad=dA$ we obtain $A(x)=xA(1)$, or $A(1)*A(1)=xA(1)$, since $x=1*1$. However, it is not hard to show that the only L^1 solutions to $g*g=xg$ are of the form $g(x)=e^{tx}$ for some t. Then $A=e^{td}$, since the two automorphisms agree on a generator of $L^1(0, 1)$. This remark has no content if $\mathscr{A}=\mathscr{A}_I$; at this writing we do not know whether this is so.

We now look more closely at the question of uniqueness for the cases when $\lambda\in\Lambda$.

Remark 7. Let $\lambda=\pm 2\pi i$ and q be a quasinilpotent derivation. Then the equation $e^{\lambda d}e^q=e^{\lambda d+q'}$ has at most one solution q', a quasinilpotent derivation. Indeed, if $e^{\lambda d+q'}=e^{\lambda d+q''}$, we see by restricting to $L^1(0, t)$ that $q'=q''$ on $L^1(0, t)$ by the proof of Theorem 14, since the spectrum of $e^{\lambda d}e^q$ does not separate the plane. Letting $t \to 1^-$ we see that $q' = q''$ on $L^1(0, 1)$.

REMARK 8. If $\lambda = \pm ib$ with $b > 2\pi$, then uniqueness does not hold. In fact, $e^{\lambda d}e^D = e^D e^{\lambda d}$ does not imply $Dd = dD$, even if $D \in \mathscr{D}'$. Indeed, let $qf = xf * \delta_{2\pi/b} = (x - 2\pi/b)f(x - 2\pi/b)$. Then q is a nilpotent derivation and belongs to $\mathscr{D}'$. By Theorem 5 or by direct computation we see that q does not commute with d, since q is not a multiple of d. However $qe^{\lambda x}f = e^{\lambda(x-2\pi/b)}(x - 2\pi/b)f(x - 2\pi/b) = e^{\lambda x}qf$, which shows that q commutes with $e^{\lambda d}$. And so we have

$$e^{\lambda d} = e^q e^{\lambda d} e^{-q} = \exp(e^q \lambda d e^{-q}),$$

but $\lambda d \neq e^q \lambda d e^{-q}$, for Corollary 15 shows that if d commutes with e^q then d must commute with q.

As a particular example let $\lambda = 2\pi i/r$ and $qf(x) = (x - r)f(x - r)$, for $1/2 \leqq r < 1$. Since $q^2 = 0$, $e^{\pm q} = 1 \pm q$ and we compute directly that $e^q d e^{-q} = d + (qd - dq) = d - rq$. Then for all t

$$e^{2\pi i d/r} = e^{tq} e^{2\pi i d/r} e^{-tq} = \exp(e^{tq}(2\pi i/r) d e^{-tq}) = \exp(2\pi i d/r - 2\pi i t q).$$

That is, for any constant c the derivations $(2\pi i/r)d$ and $(2\pi i/r)d + cq$ have the same exponential.

REMARK 9. Continuity of the logarithm. Let D be a derivation and write $D = \lambda d + q$ as usual. If $\lambda \notin \Lambda$ and if D_n are derivations such that $e^{D_n} \to e^D$, then $D_n \to D$. To see this observe by examination of (4) that we need to show only that $\lambda_n \to \lambda$, where $D_n = \lambda_n d + q_n$, since the same Γ and $\log z$ will then work for all large n. Now restrict attention to $L^1(0, \varepsilon)$, on which $\|e^D - I\| < 1$, and then the logarithmic series defines a continuous function, giving $D_n \to D$ on $L^1(0, \varepsilon)$. In particular $\lambda_n \to \lambda$, since the map $\lambda d + q \to \lambda$ is continuous. Similarly, $e^{\lambda_n d}e^{q_n} \to e^{\lambda d}e^q$ implies $\lambda_n \to \lambda$ and $q_n \to q$.

This does not hold if $\lambda \in \Lambda$. For if $|\lambda| > 2\pi$, then by Remark 8 there is a nonzero nilpotent derivation q for which $e^{\lambda d} = e^{\lambda d + q}$. For the case $\lambda = \pm 2\pi i$, let $q_n f = xf * \delta_n/(n+1)$; q_n commutes with $e^{\lambda(1+1/n)d}$, and $\|q_n\| = 1 - n/(n+1) = 1/(n+1)$. Then by Remark 8 $\exp[\lambda(1 + 1/n)d + n^2 q_n] = \exp[\lambda(1 + 1/n)d] \to e^{\lambda d}$. However,

$$\|\lambda(1 + 1/n)d + n^2 q_n\| \to \infty$$

as $n \to \infty$.

THEOREM 16. *Let λ be pure imaginary and $|\lambda| \geqq 2\pi$. Then there is a nilpotent derivation q on $L^1(0, 1)$ such that $e^{\lambda d}e^q$ is not the exponential of a derivation on $L^1(0, 1)$.*

Proof. *Case* I. $|\lambda| > 2\pi$. Let $r = 2\pi/|\lambda|$ and let the nilpotent derivation q be defined by $qf(x) = (x - r)f(x - r) = xf * \delta_r$, where δ_r is the unit mass at r. As we saw in Remark 8, $e^{\lambda d}e^q = e^q e^{\lambda d}$, since q commutes with $e^{\lambda d}$. We claim there is no derivation D such that $e^{\lambda d}e^q = e^D$. For if D exists, then D must have the form $D = \lambda d + q'$, for some quasinilpotent q'. Let μ' be the measure for which $q'f=xf*\mu'$. On $L^1(0,r)$, $q\equiv 0$ and thus $e^{\lambda d+q'}=e^{\lambda d}e^q=e^{\lambda d}$. By Remark 7, modified for $L^1(0,r)$, $\lambda d+q'=\lambda d$ on $L^1(0,r)$, and hence $\mu'=0$ on $[0,r)$.

We next show that μ' may be assumed to have no mass at r. Multiplying $e^{\lambda d}e^q=e^D$ on the left by e^{-tq} and on the right by e^{tq}, we get $e^{\lambda d}e^q=e^{-tq}e^De^{tq}=\exp(e^{-tq}De^{tq})$, since q commutes with $e^{\lambda d}$. Now $e^{-tq}De^{tq}=D+t(Dq-qD)+$ terms with at least two q's. Since $Dq-qD=\lambda(dq-qd)+(q'q-qq')=\lambda rq+(q'q-qq')$, we have $e^{-tq}De^{tq}=\lambda d+q'+\lambda rtq+Q$, where $Q=(q'q-qq')+$ terms with at least two q's. Since both q and q' involve translation by r, Q cannot contribute any mass at r. Therefore by suitable choice of t we eliminate the mass of μ' at r, since $qf=xf*\delta_r$. Henceforth we shall assume μ' has no mass at r and thus $|\mu'|[0,r]=0$.

Write $e^q=e^{-\lambda d}e^{\lambda d+q'}=1+Q'$, where Q' is a sum of terms, each of which involves q'. Let $f_k(x)=k$ for $1-r-1/k<x<1-r$ and 0 elsewhere. Apply both e^q and $1+Q'$ to f_k, integrate the results from 0 to 1, and let $k\to\infty$. Then we have $\lim\int_0^1 e^qf_k\geqq\lim\int_0^1(1+q)f_k=1+(1-r)>1$. On the other hand, since

$$|\mu'|[0,r+1/k)\to 0, \qquad \lim\int_0^1(1+Q')f_k = 1+\lim\int_0^1 Q'f_k = 1.$$

This contradiction proves Case I.

Case II. $\lambda=\pm 2\pi i$. Let $\frac{1}{2}\leqq r<1$ and let d_r be the derivation $d_rf(x)=(x-r)f(x-r)=xf*\delta_r$. By direct computation we see that $d_re^{\lambda d}=e^{\lambda d}c_rd_r$, where c_r is the constant $e^{-\lambda r}$. It follows that for all t, $e^{-td_r}e^{\lambda d}=e^{\lambda d}e^{-tc_rd_r}$, and hence

$$e^{-td_r}e^{\lambda d}e^{td_r} = e^{\lambda d}e^{t(1-c_r)d_r}. \tag{5}$$

If $e^{\lambda d}e^{d_r}=e^D$ has a solution D, then D is unique by Remark 7 and has the form $D=\lambda d+a_rd_r+q$, where a_r is a constant and $qf=xf*\mu$ for a measure μ with no mass on $[0,r]$. (The vanishing of μ on $[0,r)$ follows as before by consideration of $L^1(0,r)$, and a_r is simply the mass that was at r.) We now evaluate a_r. Multiply $e^{\lambda d}e^{d_r}=e^D$ on the left by e^{-td_r} and on the right by e^{td_r} and use (5) to obtain $e^{\lambda d}\exp[(t(1-c_r)+1)d_r]=e^{-tD_t}e^De^{td_r}=\exp(e^{-td_r}De^{td_r})$. Exactly as in Case I we obtain $e^{-td_r}De^{td_r}=\lambda d+a_rd_r+q+\lambda rtd_r+Q$, where Q contributes no mass at r. Then by choosing t so that $a_r=-\lambda rt$, we obtain the equation

$$e^{\lambda d}\exp(t(1-c_r)+1)d_r = e^{\lambda d+q'},$$

where $q'f=xf*\mu'$ for a measure μ' having no mass on $[0,r]$. As in Case I, multiply by $e^{-\lambda d}$ and use $f_k(x)=k$ on $1-r-1/k<x<1-r$, 0 elsewhere. Since $d_r^2=0$, the exponential involving d_r reduces to two terms, and after integrating and letting $k\to\infty$ we obtain $1+(1-r)[t(1-c_r)+1]=1$. Since $r\neq 1$, we find $t=(c_r-1)^{-1}$ and thus $a_r=-\lambda rt=\lambda r(1-c_r)^{-1}$.

Now let us consider the equation (E) $e^{\lambda d}e^q=e^{\lambda d+q'}$, with q and q' as variables. Let $\mathcal{Q}$ be the Banach space of derivations on $L^1(0,1)$ whose square is zero. That is,

$q \in \mathcal{Q}$ precisely when $qf = xf * \mu$ for a measure μ such that $|\mu|[0, \frac{1}{2}) = 0$. By consideration of $L^1(0, \frac{1}{2})$ as we have done previously, we see that in the equation (E), $q' \in \mathcal{Q}$ if and only if $q \in \mathcal{Q}$. Multiply (E) by $e^{-\lambda d}$, subtract I, and expand as power series in q and q', and we obtain on $\mathcal{Q}$ that $q = l(q')$, where l is linear, since all higher order terms vanish. Finding solutions to (E) when q is given in $\mathcal{Q}$ is then just inverting the linear operator l.

However, Remark 7 says that l is one-to-one, and l is clearly bounded. (In fact, by the proof of Lemma 10, $\|l(q')\| \leqq e^{2|\lambda|}\|q'\|$.) If l were onto, the closed graph theorem would give l^{-1} as a bounded linear transformation on $\mathcal{Q}$. Yet if $l^{-1}(d_r)$ exists, $l^{-1}(d_r) = a_r d_r + q = \lambda r(1-c_r)^{-1} d_r + q$, as we computed above; $l^{-1}(d_r)$ has norm at least $2\pi r|1-c_r|^{-1}(1-r)$, and $\|d_r\| = 1-r$, as we see by letting $s \to (1-r)^-$ in the formula for the norm in Theorem 2. (Recall that $qf = xf * \mu$ and $|\mu|[0, r] = 0$.) Since $2\pi r|1-c_r|^{-1} \to \infty$ as $r \to 1^-$, we find that l^{-1} does not exist. Q.E.D.

Remark 10. As a final remark we comment that the multipliers on $L^1(0, 1)$ (linear maps T: $T(f * g) = f * Tg$) are easy to describe. For each multiplier there is a finite measure μ on $[0, 1)$ such that $Tf = f * \mu = \int_0^x f(x-t)d\mu(t)$. It is clear that such a formula defines a multiplier. On the other hand, let f_n be an approximate identity for $L^1(0, 1)$ and observe that $g * Tf_n = T(g * f_n) = Tg * f_n \to Tg$ as $n \to \infty$, for each g in $L^1(0, 1)$. Since the Tf_n are uniformly bounded in L^1-norm, a subnet converges in the weak-$C(0, 1)$ topology to a measure μ. The multiplier $f \to f * \mu$ agrees with T on $C(0, 1)$ and hence on all of $L^1(0, 1)$. The mass at 1 is irrelevant, since translation by 1 annihilates $L^1(0, 1)$.

Added in proof. G. Zeller-Meier [*Sur les automorphismes des algèbres de Banach*, C. R. Acad. Sci. Paris **264** (1967), 1131–1132] has proved a stronger result than our Theorem 6. He shows that if T is a bounded automorphism of a Banach algebra and $\operatorname{sp}(T) \subset \{z \mid \operatorname{Re}(z) > 0\}$, then $T = \exp D$ for some bounded derivation D.

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